Algebra III
Semestral Exam
November 2004

Instructions. Attempt any **five** questions. The field of rational numbers is denoted by $\mathbb{Q}$.

1. (a) Let α be algebraic over a field F. Prove that $[F(\alpha):F]$, the dimension of $F(\alpha)$ as a F-vector space, equals the degree of the irreducible polynomial of α over F. (5)

   (b) Find the monic irreducible polynomial of $\sqrt[8]{7}$ over each of the following fields: $\mathbb{Q}$, $\mathbb{Q}(\sqrt{7})$, $\mathbb{Q}(\sqrt[3]{7})$. (5)

2. (a) Can -1 be written as a sum of squares in the splitting field of X^4+4 over $\mathbb{Q}$? (5)

   (b) Let f be an irreducible polynomial in $\mathbb{Q}[X]$ of degree n. Let $\phi : \mathbb{Q}[X] \to \mathbb{Q}[X]$ be any ring homomorphism. Prove that every irreducible factor of $\phi(f)$ has degree divisible by n. (5)

3. (a) Prove that the splitting field of $X^n - 1$ has degree $\phi(n)$ over $\mathbb{Q}$ ($\phi(n)$ denotes the number of integers which are less than n and are co-prime to n). (5)

   (b) Let $m > 1$ be an integer. Prove that there are infinitely many degree m Galois extensions of $\mathbb{Q}$ with cyclic Galois group. (5)

4. (a) Let f be an irreducible quartic polynomial over $\mathbb{Q}$ with exactly two real roots. What can you say about its Galois group over $\mathbb{Q}$? (5)

   (b) Let F be a field of characteristic zero and let $a \in F$. Suppose that the polynomial $x^p - a$ is reducible in $F[x]$ for some prime p. Prove that a has a pth root in F. (5)

5. (a) Let $\alpha = \sqrt[3]{a + b\sqrt{2}}$ with $a, b \in \mathbb{Q}$. Let ξ denote the primitive cube root of unity and f be the irreducible polynomial of α over $\mathbb{Q}(\xi)$. Determine the possible Galois groups of f over $\mathbb{Q}(\xi)$. (6)

   (b) Let F be a finite field and $f \in F[X]$ be a non-constant polynomial whose derivative is zero. Prove that f is not irreducible in $F[X]$. (4)

6. (a) Let K be the splitting field of the polynomial X^3+X+1 over $\mathbb{Q}$. Determine the elements of K whose cube is a rational number. (5)

   (b) Let K_1 and K_2 be two Galois extensions of $\mathbb{Q}$ with Galois groups G_1 and G_2 respectively. Show that K_1K_2 is a Galois extension of $\mathbb{Q}$ and determine its Galois group. (5)